

Aspects of black hole entropy

Shinji Mukohyama ¹

*Department of Physics and Astronomy, University of Victoria
Victoria, BC, Canada V8W 3P6*

and

*Canadian Institute for Theoretical Astrophysics, University of Toronto
Toronto, ON, M5S 3H8*

Abstract

There have been many attempts to understand the statistical origin of black-hole entropy. Among them, entanglement entropy and the brick wall model are strong candidates. In this paper, first, we show that the entanglement approach reduces to the brick wall model when we seek the maximal entanglement entropy. After that, the stability of the brick wall model is analyzed in a rotating background. It is shown that in the Kerr background without horizon but with an inner boundary a scalar field has complex-frequency modes and that, however, the imaginary part of the complex frequency can be small enough compared with the Hawking temperature if the inner boundary is sufficiently close to the horizon, say at a proper altitude of Planck scale. Hence, the brick wall model is well defined even in a rotating background if the inner boundary is sufficiently close to the horizon. These results strongly suggest that the entanglement approach is also well defined in a rotating background.

1 Introduction

Black hole entropy is given by a mysterious formula called the Bekenstein-Hawking formula [1, 2]: $S_{BH} = A/4l_{pl}^2$, where A is area of the horizon. There have been many attempts to understand the statistical origin of the black-hole entropy [3].

Entanglement entropy [4, 5] is one of the strongest candidates of the origin of black hole entropy. It is originated from a direct-sum structure of a Hilbert space of a quantum system: for an element $|\psi\rangle$ of the Hilbert space \mathcal{F} of the form

$$\mathcal{F} = \mathcal{F}_I \bar{\otimes} \mathcal{F}_{II}, \quad (1)$$

the entanglement entropy S_{ent} is defined by

$$S_{ent} = -\text{Tr}_I[\rho_I \ln \rho_I], \quad \rho_I = \text{Tr}_{II}|\psi\rangle\langle\psi|. \quad (2)$$

Here $\bar{\otimes}$ denotes a tensor product followed by a suitable completion and $\text{Tr}_{I,II}$ denotes a partial trace over $\mathcal{F}_{I,II}$, respectively.

On the other hand, there is another strong candidate for the origin of black hole entropy: the brick wall model introduced by 'tHooft [6]. In this model, thermal atmosphere in equilibrium with a black hole is considered. In this situation, we encounter with two kinds of divergences in physical quantities. The first is due to infinite volume of the system and the second is due to infinite blue shift near the horizon. We are not interested in the first since it represents contribution from matter in the far distance. Hence we introduce an outer boundary in order to make our system finite. It is the second divergence that we would like to associate with black hole entropy. Namely, it can be shown by introducing a Planck scale cutoff that entropy of the thermal atmosphere near the horizon is proportional to the area of the horizon in Planck units.

In this paper, first, we show that the brick wall model seeks the maximal value of the entanglement entropy. In other words, the entanglement approach reduces to the brick wall model when we seek the maximal value of the entanglement entropy. After that, we analyze the stability of the brick wall model in a rotating background. We show that the time scale of the ergoregion instability is much longer than

¹E-mail: mukoyama@uvic.ca

the relaxation time scale of the thermal state with the Hawking temperature. In the latter time scale ambient fields should settle in the thermal state. Thus, the brick wall model is well defined even in a rotating background. These results strongly suggest that the entanglement approach is also well defined in a rotating background.

This paper is organized as follows. In Sec. 2 the relation between the entanglement entropy and the brick wall model is shown. In Sec. 3 the stability of the brick wall model is analyzed in the Kerr background. Sec. 4 is devoted to a summary of this paper.

2 Brick wall entropy as maximal entanglement entropy

For simplicity, we consider a minimally coupled, real scalar field described by the action

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} [g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \mu^2 \phi^2], \quad (3)$$

in a spherically symmetric, static black-hole space-time

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2. \quad (4)$$

We denote the area radius of the horizon by r_0 and the surface gravity by κ_0 ($\neq 0$):

$$f(r_0) = 0, \quad \kappa_0 = \frac{1}{2} f'(r_0). \quad (5)$$

We quantize the system of the scalar field with respect to the Killing time t in a Kruskal-like extension of the black hole spacetime. The corresponding ground state is called the Boulware state and its energy density is known to diverge near the horizon. Although we shall only consider states with bounded energy density, it is convenient to express these states as excited states above the Boulware ground state for technical reasons. Hence, we would like to introduce an ultraviolet cutoff α with dimension of length to control the divergence. The cutoff parameter α is implemented so that we only consider two regions satisfying $r > r_1$ (shaded regions *I* and *II* in *Figure 1*), where r_1 ($> r_0$) is determined by

$$\alpha = \int_{r_0}^{r_1} \frac{dr}{\sqrt{f(r)}}. \quad (6)$$

[Evidently, the limit $\alpha \rightarrow 0$ corresponds to the limit $r_1 \rightarrow r_0$. Thus, in this limit, the whole region in which $\partial/\partial t$ is timelike is covered.] Strictly speaking, we also have to introduce outer boundaries, say at $r = L$ ($\gg r_0$), to control the infinite volume of the constant- t surface. However, even if there are outer boundaries, the following arguments still hold.

In this situation, there is a natural choice for division of the system of the scalar field: let \mathcal{H}_I be the space of mode functions with supports in the region *I* and \mathcal{H}_{II} be the space of mode functions with supports in the region *II*. Thence, the space \mathcal{F} of all states are of the form (1), where \mathcal{F}_I and \mathcal{F}_{II} are defined as symmetric Fock spaces constructed from \mathcal{H}_I and \mathcal{H}_{II} , respectively:

$$\mathcal{F}_{I,II} \equiv \mathbb{C} \oplus \mathcal{H}_{I,II} \oplus (\mathcal{H}_{I,II} \bar{\otimes} \mathcal{H}_{I,II})_{sym} \oplus \cdots. \quad (7)$$

Here $(\cdots)_{sym}$ denotes the symmetrization.

Let us investigate what kind of condition should be imposed for our arguments to be self-consistent. A clear condition is that the backreaction of the scalar field to the background geometry should be finite. For the brick wall model this condition is satisfied. Namely, in Ref. [7], it was shown that the total mass of the thermal atmosphere of quantum fields is actually bounded. Thus, also for our system, we would like to impose the condition that the contribution ΔM of the subsystem \mathcal{F}_I to the mass of the geometry should be bounded in the limit $\alpha \rightarrow 0$.

It is easily shown that ΔM is given by

$$\Delta M \equiv - \int_{x \in I} T_t^t 4\pi r^2 dr = H_I, \quad (8)$$

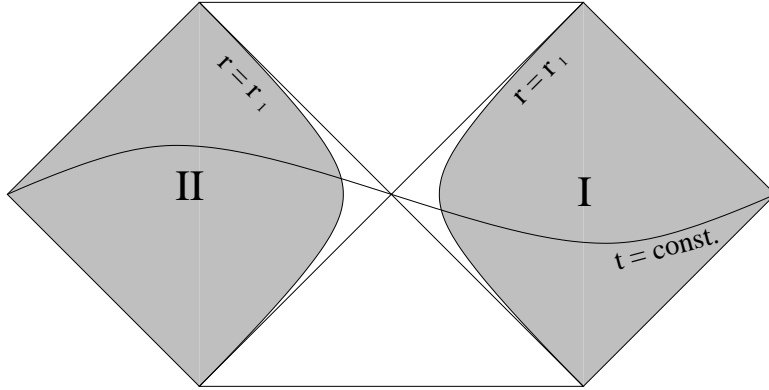


Figure 1: The Kruskal-like extension of the static, spherically symmetric black-hole spacetime. We consider only the regions satisfying $r > r_1$ (the shaded regions *I* and *II*).

where H_I is the Hamiltonian of the subsystem \mathcal{F}_I with respect to the Killing time t . Hence, the expectation value of ΔM with respect to a state $|\psi\rangle$ of the scalar field is decomposed into the contribution of excitations and the contribution from the zero-point energy:

$$\langle\psi|\Delta M|\psi\rangle = E_{ent} + \Delta M_B, \quad (9)$$

where E_{ent} is entanglement energy defined by

$$E_{ent} \equiv \langle\psi| : H_I : |\psi\rangle, \quad (10)$$

and ΔM_B is the zero-point energy of the Boulware state. Here, the colons denote the usual normal ordering. [This definition of entanglement energy corresponds to $E_{ent}^{(I')}$ in Ref. [8] and $\langle : H_2 : \rangle$ in Ref. [3].]

Since the Boulware energy ΔM_B diverges as $\Delta M_B \sim -AT_H\alpha^{-2}$ in the limit $\alpha \rightarrow 0$ [7], we should impose the condition

$$E_{ent} \simeq |\Delta M_B|, \quad (11)$$

where $A = 4\pi r_0^2$ is the area of the horizon, $T_H = \kappa_0/2\pi$ is the Hawking temperature. We call this condition *the small backreaction condition (SBC)*. Note that the right hand side of SBC (11) is independent of the state $|\psi\rangle$.

Now, we shall show that the Hartle-Hawking state is a maximum of the entanglement entropy in the space of quantum states satisfying SBC. For this purpose, we have a more general statement for a quantum system with a state-space of the form (1): *a state of the form*

$$|\psi\rangle = \mathcal{N} \sum_n e^{-E_n/2T} |n\rangle_I \otimes |n\rangle_{II} \quad (12)$$

is a maximum of the entanglement entropy in the space of states with fixed expectation value of the operator E_I defined by

$$E_I = \left(\sum_n E_n |n\rangle_I \cdot {}_I\langle n| \right) \otimes \left(\sum_m |m\rangle_{II} \cdot {}_{II}\langle m| \right), \quad (13)$$

provided that the real constant T is determined so that the expectation value of E_I is actually the fixed value. Here, $\{|n\rangle_I\}$ and $\{|n\rangle_{II}\}$ ($n = 1, 2, \dots$) are bases of the subspaces \mathcal{F}_I and \mathcal{F}_{II} , respectively, and E_n are assumed to be real and non-negative. (For a proof of this statement, see Ref. [9].) Note that this statement is almost the same as the following statement in statistical mechanics: a canonical state is a maximum of statistical entropy in the space of states with fixed energy, provided that the temperature of the canonical state is determined so that the energy is actually the fixed value.

Note that the expectation value of E_I is equal to the entanglement energy (10), providing that $|n\rangle_I$ and E_n are an eigenstate and an eigenvalue of the normal-ordered Hamiltonian : H_I : of the subsystem \mathcal{F}_I . Hence, for the system of the scalar field, the above general statement insists that the state (12) is a maximum of the entanglement entropy in the space of states satisfying SBC, which corresponds to fixing the entanglement entropy. Of course, in this case, the constant T should be determined so that SBC (11) is satisfied. The value of T is easily determined as $T = T_H$ by using the well-known fact that the negative divergence in the Boulware energy density can be canceled by thermal excitations if and only if temperature with respect to the time t is equal to the Hawking temperature.

Finally, we obtain the statement that the Hartle-Hawking state [11] is a maximum of entanglement entropy in the space of quantum states satisfying SBC since the Hartle-Hawking state is actually of the form (12) with $T = T_H$ [12]. [Strictly speaking, in order to obtain the Hartle-Hawking state, we have to take the limit $\alpha \rightarrow 0$ (and $L \rightarrow \infty$). However, the following arguments still hold for a finite value of α (and L).] The corresponding reduced density matrix is the thermal state with temperature equal to the Hawking temperature. Therefore, the maximal entanglement entropy is equal to the thermal entropy with the Hawking temperature, which is sought in the brick wall model.

In summary the brick wall model seeks the maximal value of entanglement entropy [9, 14]. In other words, the entanglement approach reduces to the brick wall model when we seek the maximal entanglement entropy.

3 Stability of the brick wall model in a rotating background

Originally, the brick wall model was proposed in a spherically symmetric, static background, say, the Schwarzschild background. Hence, it seems interesting to see how this model is extended to a rotating background, say, the Kerr background [15, 16, 17, 18]. However, it is known that in a rapidly rotating spacetime without horizon a field has complex-frequency modes [19, 20] and that there is the so called ergoregion instability [21]. Thus, it might be expected that the brick-wall model in rotating background might be unstable and unsuitable for the origin of black hole entropy.

For simplicity, let us consider the Kerr spacetime as a background. The metric is given by

$$ds^2 = - \left(1 - \frac{2Mr}{\Sigma} \right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + R^2 \sin^2 \theta d\varphi^2 - \frac{4Mar}{\Sigma} \sin^2 \theta d\varphi dt, \quad (14)$$

where

$$\begin{aligned} \Sigma &= r^2 + a^2 \cos^2 \theta, \\ \Delta &= r^2 + a^2 - 2Mr, \\ \Sigma R^2 &= (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta. \end{aligned} \quad (15)$$

We only consider the region $r \geq r_1$ in this spacetime: we impose the Dirichlet boundary condition on the field ϕ at $r = r_1$. (For more general boundary conditions, see Ref. [22].) Hereafter we assume that $r_1 > M + \sqrt{M^2 - a^2}$: there is no horizon in the region $r \geq r_1$.

It is well known that in this background the field equation of a massless real (Klein-Gordon) scalar field becomes separable. In fact, we can find a solution of the field equation of the form

$$f = \frac{u(r)}{\sqrt{r^2 + a^2}} S(\theta), \quad (16)$$

where S satisfies

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dS}{d\theta} \right) + \left[\lambda - a^2 \mu^2 (y^2 - 1) \sin^2 \theta - \frac{m^2}{\sin^2 \theta} \right] S = 0, \quad (17)$$

and the equation for u can be written as

$$\frac{d^2 u}{dx^2} + (y - V_+)(y - V_-)u = 0 \quad (18)$$

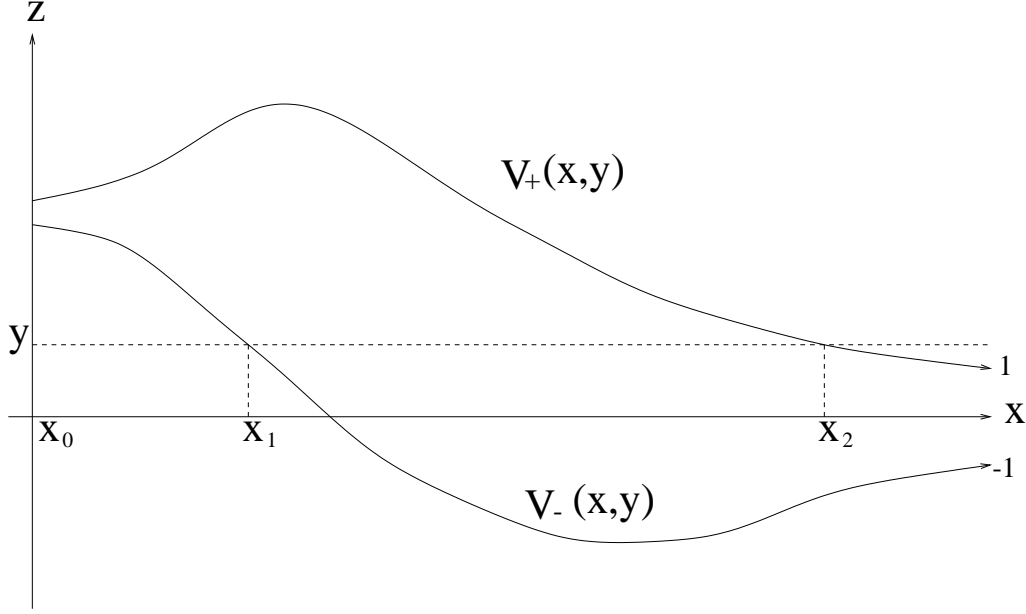


Figure 2: The typical form of the graphs $z = V_{\pm}(x, y)$ is written on a fixed y plane. Note that V_{\pm} depend on y as well as x through the eigen value λ of equation (17). However, asymptotic behavior of V_{\pm} in the limit $x \rightarrow \pm\infty$ does not depend on y .

by introducing the non-dimensional tortoise coordinate x by

$$\mu^{-1} \frac{dx}{dr} = \frac{r^2 + a^2}{\Delta}, \quad (19)$$

or

$$\mu^{-1} x = r + \frac{M}{\sqrt{M^2 - a^2}} \left[r_+ \ln \left(\frac{r - r_+}{r_+ - r_-} \right) - r_- \ln \left(\frac{r - r_-}{r_+ - r_-} \right) \right]. \quad (20)$$

Here, μ is the mass of the scalar field, $r_{\pm} = M \pm \sqrt{M^2 - a^2}$, $y \equiv \omega/\mu$ and V_{\pm} are defined by

$$\begin{aligned} \mu V_{\pm} &= \frac{2maMr}{(r^2 + a^2)^2} \pm \sqrt{\frac{\Delta \tilde{\mu}^2}{r^2 + a^2}}, \\ \tilde{\mu}^2 &= \mu^2 + \frac{\lambda}{(r^2 + a^2)} - \frac{m^2 a^2 (r^2 + a^2 + 2Mr)}{(r^2 + a^2)^3} + \frac{2Mr + a^2}{(r^2 + a^2)^2} - \frac{6Ma^2 r}{(r^2 + a^2)^3}. \end{aligned} \quad (21)$$

Note that in the horizon limit $r \rightarrow r_+$ (or $x \rightarrow -\infty$) both of V_{\pm} approach to the same value

$$V_{\pm} \rightarrow \frac{ma}{2\mu M r_+}, \quad (22)$$

and that V_{\pm} approaches to ± 1 , respectively, in the limit $r \rightarrow \infty$ (or $x \rightarrow \infty$). (See Fig. 2.)

In Ref. [23], the existence of complex frequency modes was shown by examining a scattering amplitude for real-frequency waves. The strategy is based on the following expected form of the scattering amplitude S near a pole $y = y_R + iy_I$, providing that $|y_R| \gg |y_I|$ [20].

$$S = e^{2i\delta_0} \times \frac{y - y_R + iy_I}{y - y_R - iy_I}, \quad (23)$$

where δ_0 is a constant phase. Hence, if we can obtain this form of a scattering amplitude by analyzing real-frequency waves then we find an outgoing normal mode corresponding to $y = y_R + iy_I$ and an

incoming normal mode corresponding to $y = y_R - iy_I$. After that we should investigate whether these normal modes converge or diverge in the limit of $x \rightarrow \infty$. If these normal modes converge then they give complex-frequency mode functions.

By analyzing Eq. (18) in the WKB approximation and using the above strategy, we can obtain the following results [23].

- (a) When $ma > 2\mu Mr_+$ and $1 < y < V_-(x_1)$, we obtain a set of complex-frequency modes. However, the imaginary part $\Im\omega$ of the complex frequency is small enough:

$$T_{BH}^{-1}\Im\omega \sim \pm\pi e^{-2\eta} \left| \ln \left(\frac{r_1 - r_+}{r_+ - r_-} \right) \right|^{-1} \rightarrow 0 \quad (r_1 \rightarrow r_+), \quad (24)$$

where T_{BH} is the Hawking temperature of the Kerr background.

- (b) When $ma < -2\mu Mr_+$ and $V_+(x_1) < y < -1$, we obtain a set of complex-frequency modes. However, the imaginary part $\Im\omega$ of the complex frequency is small enough:

$$T_{BH}^{-1}\Im\omega \sim \pm\pi e^{-2\eta} \left| \ln \left(\frac{r_1 - r_+}{r_+ - r_-} \right) \right|^{-1} \rightarrow 0 \quad (r_1 \rightarrow r_+). \quad (25)$$

- (c) In other cases, there is no complex-frequency modes.

In this section we have shown that in the Kerr background without horizon but with an inner boundary a scalar field has complex-frequency modes and that the imaginary part of the complex frequency is small enough compared with the Hawking temperature if the inner boundary is sufficiently close to the horizon, say at a proper altitude of Planck scale. Hence, the time scale of the ergoregion instability is much longer than the relaxation time scale of the thermal state with the Hawking temperature. In the latter time scale ambient fields should settle in the thermal state. In this sense ergoregion instability is not so catastrophic. Thus, the brick wall model is well defined if the inner boundary is sufficiently close to the horizon.

4 Summary and discussion

In this paper we have analyzed the entanglement entropy and the brick wall model. In Sec. 2 we have shown that the entanglement approach reduces to the brick wall model when we seek the maximal value of the entanglement entropy. In Sec. 3 we have shown that the brick wall model is well defined even in a rotating background. These two results strongly suggest that the entanglement approach is also well defined in a rotating background.

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